The Vasicek Interest Rate Process
Part II - The Stochastic Discount Rate

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March, 2013

In Part I of this series on the Vasicek Interest Rate Process we determined that the equation for the random short rate at time t (r_t) given the short rate at time s (r_s) was... [1]

$$r_t = r_\infty + \mathrm{Exp}\left\{-\lambda\,(t-s)\right\}(r_s - r_\infty) + \sigma\,\mathrm{Exp}\left\{-\lambda\,t\right\}\int\limits_s^t \mathrm{Exp}\left\{\lambda\,u\right\}\delta W_u \tag{1}$$

In Equation (1) above r_s is the short rate at time s, r_∞ is the long-term short rate mean, λ is the rate of mean reversion, σ is the annualized short rate volatility, and δW_u is the change in the driving Brownian motion over the infinitesimally small time interval $[u, u + \delta u]$.

The price of a pure discount bond ($P(s,t)$) at time s may be written as the expectation of the path integral of the short rate over the time interval $[s,t]$. The price at time s of a pure discount bond that matures at time t and pays one dollar at maturity given the information set available to us at time s (I_s) can be written as...

$$P(s,t) = \mathbb{E}\left[\mathrm{Exp}\left\{-\int\limits_s^t r_u\,\delta u\right\}\,\middle|\, I_s\right] \tag{2}$$

Note that the solution to the integral in bond pricing Equation (2) above is the average short rate of interest over the time interval $[s,t]$. This average interest rate is referred to as the stochastic discount rate. In this white paper we will develop equations for the mean and variance of the stochastic discount rate.

Our Hypothetical Problem

The go-forward interest rate assumptions from Part I were...

Description	Symbol	Value
Current short rate	r_s	0.04
Long-term short rate mean	r_∞	0.09
Annualized short rate volatility	σ	0.03
Mean reversion rate	λ	0.35

Question 1: What is the expected stochastic discount rate mean over the time interval $[0,10]$?

Question 2: What is the expected stochastic discount rate variance over the time interval $[0,10]$?

The Stochastic Discount Rate

We will define the variable $R_{s,t}$ to be the stochastic discount rate applicable to the time interval $[s,t]$. Using Equation (2) above the equation for the stochastic discount rate is...

$$R_{s,t} = \int\limits_s^t r_u\,\delta u \tag{3}$$

Using Equations (1) and (3) above the equation for the stochastic discount rate $R_{s,t}$ becomes...

$$R_{s,t} = \int\limits_s^t \bigg(r_\infty + \text{Exp}\bigg\{-\lambda\,(t-s)\bigg\}(r_s - r_\infty) + \sigma\,\text{Exp}\bigg\{-\lambda\,t\bigg\}\int\limits_s^t \text{Exp}\bigg\{\lambda\,v\bigg\}\,\delta W_v\bigg)\delta u$$

$$= r_\infty \int\limits_s^t \delta u + (r_s - r_\infty)\int\limits_s^t \text{Exp}\bigg\{-\lambda\,(u-s)\bigg\}\,\delta u + \int\limits_s^t \bigg(\sigma\,\text{Exp}\bigg\{-\lambda\,u\bigg\}\int\limits_s^t \text{Exp}\bigg\{\lambda\,v\bigg\}\,\delta W_v\bigg)\delta u \tag{4}$$

We will make to following integral definitions...

$$I_1 = r_\infty \int\limits_s^t \delta u \;\Bigg|\; I_2 = (r_u - r_\infty)\int\limits_s^t \text{Exp}\bigg\{-\lambda\,(u-s)\bigg\}\,\delta u \;\Bigg|\; I_3 = \int\limits_s^t \bigg(\sigma\,\text{Exp}\bigg\{-\lambda\,u\bigg\}\int\limits_s^t \text{Exp}\bigg\{\lambda\,v\bigg\}\,\delta W_v\bigg)\delta u \tag{5}$$

Using Equation (5) above we can rewrite stochastic discount rate Equation (4) above as...

$$R_{s,t} = I_1 + I_2 + I_3 \tag{6}$$

Using Appendix Equations (24), (26) and (27) below we can rewrite Equation (6) above as...

$$R_{s,t} = r_\infty\,(t-s) + (r_\infty - r_s)\bigg(\text{Exp}\bigg\{-\lambda\,(t-s)\bigg\} - 1\bigg)\,\lambda^{-1} + \sigma\int\limits_s^t\int\limits_s^t \text{Exp}\bigg\{-\lambda\,(u-v)\bigg\}\,\delta W_v\,\delta u \tag{7}$$

The First Moment of the Distribution of the Stochastic Discount Rate

Using Equation (3) above the equation for the first moment of the distribution of the stochastic discount rate is...

$$\mathbb{E}\bigg[R_{s,t}\bigg] = \mathbb{E}\bigg[\int\limits_s^t r_u\,\delta u\bigg] = \int\limits_s^t \mathbb{E}\bigg[r_u\bigg]\,\delta u \tag{8}$$

Using Equation (7) above we can rewrite Equation (8) above as...

$$\mathbb{E}\bigg[R_{s,t}\bigg] = \mathbb{E}\bigg[r_\infty\,(t-s) + (r_\infty - r_s)\bigg(\text{Exp}\bigg\{-\lambda\,(t-s)\bigg\} - 1\bigg)\,\lambda^{-1} + \sigma\int\limits_s^t\int\limits_s^t \text{Exp}\bigg\{-\lambda\,(u-v)\bigg\}\,\delta W_v\,\delta u\bigg] \tag{9}$$

Given that the right hand side of Equation (9) above is random we can rewrite that expectation as...

$$\mathbb{E}\bigg[R_{s,t}\bigg] = r_\infty\,(t-s) + (r_\infty - r_s)\bigg(\text{Exp}\bigg\{-\lambda\,(t-s)\bigg\} - 1\bigg)\,\lambda^{-1} + \sigma\int\limits_s^t\int\limits_s^t \text{Exp}\bigg\{-\lambda\,(u-v)\bigg\}\,\mathbb{E}\bigg[\delta W_v\,\delta u\bigg] \tag{10}$$

Note the following expectation...

$$\mathbb{E}\bigg[\delta W_v\,\delta u\bigg] = 0 \tag{11}$$

Using Equation (11) above the solution to Equation (10) above is...

$$\mathbb{E}\bigg[R_{s,t}\bigg] = r_\infty\,(t-s) + (r_\infty - r_s)\bigg(\text{Exp}\bigg\{-\lambda\,(t-s)\bigg\} - 1\bigg)\,\lambda^{-1} \tag{12}$$

The Second Moment of the Distribution of the Stochastic Discount Rate

Using Equation (3) above the equation for the second moment of the distribution of the stochastic discount rate is...

$$\mathbb{E}\bigg[R^2_{s,t}\bigg] = \mathbb{E}\bigg[\bigg(\int\limits_s^t r_u\,\delta u\bigg)^2\bigg] = \mathbb{E}\bigg[\int\limits_s^t\int\limits_s^t r_u\,r_v\,\delta u\,\delta v\bigg] = \int\limits_s^t\int\limits_s^t \mathbb{E}\bigg[r_u\,r_v\bigg]\,\delta u\,\delta v \tag{13}$$

Given that we are currently standing at time t and r_t is the short rate at time t (known), using Equation (3) above the equations for the short rate at future time u (random) and the short rate at future time v (random) are...

$$r_u = r_t + \int_t^u \delta r_x \text{ ...and... } r_v = r_t + \int_t^v \delta r_y \tag{14}$$

Using Equations (13) and (14) above the equation for the expected product of the short rate at future time u (random) and the short rate at future time v from Part I of this series is... [1]

$$\mathbb{E}\bigg[r_u\, r_v\bigg] = \mathbb{E}\bigg[r_u\bigg]\mathbb{E}\bigg[r_v\bigg] + \frac{1}{2}\sigma^2 \operatorname{Exp}\bigg\{-\lambda\,(u+v)\bigg\}\bigg(\operatorname{Exp}\bigg\{2\,\lambda\,(u\wedge v)\bigg\} - \operatorname{Exp}\bigg\{2\,\lambda\,t\bigg\}\bigg)\lambda^{-1} \tag{15}$$

Using Equation (15) above we can rewrite Equation (13) as...

$$\begin{aligned}
\mathbb{E}\bigg[R_{s,t}^2\bigg] &= \int_s^t\int_s^t \mathbb{E}\bigg[R_u\bigg]\mathbb{E}\bigg[R_v\bigg]\,\delta u\,\delta v + \frac{\sigma^2}{2\,\lambda}\int_s^t\int_s^t \operatorname{Exp}\bigg\{-\lambda\,(u+v)\bigg\}\bigg(\operatorname{Exp}\bigg\{2\,\lambda\,(u\wedge v)\bigg\} - \operatorname{Exp}\bigg\{2\,\lambda\,t\bigg\}\bigg)\delta u\,\delta v \\
&= \int_s^t \mathbb{E}\bigg[R_u\bigg]\,\delta u \int_s^t \mathbb{E}\bigg[R_v\bigg]\,\delta v + \frac{\sigma^2}{2\,\lambda}\int_s^t\int_s^t \operatorname{Exp}\bigg\{-\lambda\,(u+v)\bigg\}\bigg(\operatorname{Exp}\bigg\{2\,\lambda\,(u\wedge v)\bigg\} - \operatorname{Exp}\bigg\{2\,\lambda\,t\bigg\}\bigg)\delta u\,\delta v \\
&= \bigg[\mathbb{E}\bigg[R_{s,t}\bigg]\bigg]^2 + \frac{\sigma^2}{2\,\lambda}\bigg(\int_s^t\int_s^t \operatorname{Exp}\bigg\{-\lambda\,(u+v-2\,(u\wedge v))\bigg\}\,\delta u\,\delta v - \int_s^t\int_s^t \operatorname{Exp}\bigg\{-\lambda\,(u+v)+2\,\lambda\,s\bigg\}\,\delta u\,\delta v\bigg)
\end{aligned} \tag{16}$$

We will make the following integral definitions...

$$I_4 = \int_s^t\int_s^t \operatorname{Exp}\bigg\{-\lambda\,(u+v-2\,(u\wedge v))\bigg\}\,\delta u\,\delta v \text{ ...and... } I_5 = \int_s^t\int_s^t \operatorname{Exp}\bigg\{-\lambda\,(u+v)\bigg\}\,\delta u\,\delta v \tag{17}$$

Using the integral definitions in Equation (17) above we can rewrite Equation (16) above as...

$$\mathbb{E}\bigg[R_{s,t}^2\bigg] = \bigg[\mathbb{E}\bigg[R_{s,t}\bigg]\bigg]^2 + \frac{\sigma^2}{2\,\lambda}\bigg(I_4 - \operatorname{Exp}\bigg\{2\,\lambda\,s\bigg\}\,I_5\bigg) \tag{18}$$

Using Appendix Equations (29), (35) and (38) below the solution to Equation (18) above is...

$$\begin{aligned}
\mathbb{E}\bigg[R_{s,t}^2\bigg] &= \bigg[\mathbb{E}\bigg[R_{s,t}\bigg]\bigg]^2 + \frac{\sigma^2}{2\,\lambda}\bigg(\frac{2}{\lambda}\bigg[(t-s) + \frac{1}{\lambda}\bigg(\operatorname{Exp}\bigg\{-\lambda\,(t-s)\bigg\} - 1\bigg)\bigg] \\
&\quad - \frac{1}{\lambda^2}\bigg[1 - 2\operatorname{Exp}\bigg\{-\lambda\,(t-s)\bigg\} + \operatorname{Exp}\bigg\{-2\,\lambda\,(t-s)\bigg\}\bigg]\bigg) \\
&= \bigg[\mathbb{E}\bigg[R_{s,t}\bigg]\bigg]^2 + \frac{\sigma^2}{2\,\lambda}\bigg(\frac{2}{\lambda}(t-s) + \frac{2}{\lambda^2}\operatorname{Exp}\bigg\{-\lambda\,(t-s)\bigg\} - \frac{2}{\lambda^2} - \frac{1}{\lambda^2} \\
&\quad + \frac{2}{\lambda^2}\operatorname{Exp}\bigg\{-\lambda\,(t-s)\bigg\} - \frac{1}{\lambda^2}\operatorname{Exp}\bigg\{-2\,\lambda\,(t-s)\bigg\}\bigg) \\
&= \bigg[\mathbb{E}\bigg[R_{s,t}\bigg]\bigg]^2 + \frac{\sigma^2}{2\,\lambda}\bigg(\frac{2}{\lambda}(t-s) + \frac{4}{\lambda^2}\operatorname{Exp}\bigg\{-\lambda\,(t-s)\bigg\} - \frac{3}{\lambda^2} - \frac{1}{\lambda^2}\operatorname{Exp}\bigg\{-2\,\lambda\,(t-s)\bigg\}\bigg) \\
&= \bigg[\mathbb{E}\bigg[R_{s,t}\bigg]\bigg]^2 + \frac{\sigma^2}{2\,\lambda^3}\bigg(2\,\lambda\,(t-s) - 3 + 4\operatorname{Exp}\bigg\{-\lambda\,(t-s)\bigg\} - \operatorname{Exp}\bigg\{-2\,\lambda\,(t-s)\bigg\}\bigg)
\end{aligned} \tag{19}$$

The Stochastic Discount Rate Mean and Variance

Using Equation (12) above the equation for the stochastic discount rate mean is...

$$\text{mean} = \mathbb{E}\bigg[R_{s,t}\bigg] = r_\infty\,(t-s) + (r_\infty - r_s)\bigg(\operatorname{Exp}\bigg\{-\lambda\,(t-s)\bigg\} - 1\bigg)\lambda^{-1} \tag{20}$$

Using Equations (12) and (19) above the equation for the stochastic discount rate variance is...

$$\text{variance} = \frac{\sigma^2}{2\,\lambda^3}\bigg(2\,\lambda\,(t-s) - 3 + 4\,\text{Exp}\bigg\{-\lambda\,(t-s)\bigg\} - \text{Exp}\bigg\{-2\,\lambda\,(t-s)\bigg\}\bigg) \tag{21}$$

Answers To Our Hypothetical Problem

Question 1: What is the expected stochastic discount rate mean over the time interval $[0, 10]$?

Using Equation (20) and the assumptions table above the expected stochastic discount rate over the time interval $[0, 10]$ is...

$$\begin{aligned} \text{mean} &= 0.09 \times (10-0) + (0.09 - 0.04) \times \bigg(\text{Exp}\bigg\{-0.35 \times (10-0)\bigg\} - 1\bigg) \times 0.35^{-1} \\ &= 0.76146 \end{aligned} \tag{22}$$

Question 2: What is the expected stochastic discount rate variance over the time interval $[0, 10]$?

Using Equation (21) and the assumptions table above the variance of the expected stochastic discount rate over the time interval $[0, 10]$ is...

$$\begin{aligned} \text{variance} &= \frac{0.03^2}{2 \times 0.35^3} \times \bigg(2 \times 0.35 \times (10-0) - 3 + 4 \times \text{Exp}\bigg\{-0.35 \times (10-0)\bigg\} - \text{Exp}\bigg\{-2 \times 0.35 \times (10-0)\bigg\}\bigg) \\ &= 0.04324 \end{aligned} \tag{23}$$

Appendix

A. The solution to integral I_1 in Equations (5) and (6) above is...

$$I_1 = r_\infty \int_s^t \delta u = r_\infty\,(t-s) \tag{24}$$

B. The solution to integral I_2 in Equations (5) and (6) above is...

$$I_2 = (r_s - r_\infty) \int_s^t \text{Exp}\bigg\{-\lambda\,(u-s)\bigg\}\,\delta u = (r_s - r_\infty)\,\text{Exp}\bigg\{\lambda\,s\bigg\} \int_s^t \text{Exp}\bigg\{-\lambda\,u\bigg\}\,\delta u \tag{25}$$

After solving the integral in Equation (25) above that equation becomes...

$$I_2 = -\frac{1}{\lambda}\,(r_s - r_\infty)\,\text{Exp}\bigg\{\lambda\,s\bigg\}\bigg(\text{Exp}\bigg\{-\lambda\,t\bigg\} - \text{Exp}\bigg\{-\lambda\,s\bigg\}\bigg) = (r_\infty - r_s)\,\bigg(\text{Exp}\bigg\{-\lambda\,(t-s)\bigg\} - 1\bigg)\,\lambda^{-1} \tag{26}$$

C. The solution to integral I_3 in Equations (5) and (6) above as...

$$I_3 = \int_s^t \bigg(\sigma\,\text{Exp}\bigg\{-\lambda\,u\bigg\} \int_s^t \text{Exp}\bigg\{\lambda\,v\bigg\}\,\delta W_v\bigg)\,\delta u = \sigma \int_s^t \int_s^t \text{Exp}\bigg\{-\lambda\,(u-v)\bigg\}\,\delta W_v\,\delta u \tag{27}$$

D. We want to solve the following integral from Equations (17) and (18) above...

$$I_4 = \int_s^t \int_s^t \text{Exp}\bigg\{-\lambda\,(u + v - 2\,(u \wedge v))\bigg\}\,\delta u\,\delta v \tag{28}$$

The solution to the integral in Equation (28) above is... [2]

$$I_4 = \frac{2}{\lambda}\bigg[(t-s) + \frac{1}{\lambda}\bigg(\text{Exp}\bigg\{-\lambda\,(t-s)\bigg\} - 1\bigg)\bigg] \tag{29}$$

E. We want to solve the following integral from Equations (17) and (18) above...

$$I_5 = \int_s^t \int_s^t \mathrm{Exp}\left\{ -\lambda\,(u+v) \right\} \delta u\, \delta v \tag{30}$$

The soluton to the inner integral in Equation (30) above is...

$$\int_s^t \mathrm{Exp}\left\{ -\lambda\,(u+v) \right\} \delta u = -\frac{1}{\lambda}\,\mathrm{Exp}\left\{ -\lambda\,(u+v) \right\} \Bigg[_{u=s}^{u=t} = \frac{1}{\lambda}\left(\mathrm{Exp}\left\{ -\lambda\,(s+v) \right\} - \mathrm{Exp}\left\{ -\lambda\,(t+v) \right\} \right) \tag{31}$$

Using Equation (31) above we can rewrite Equation (30) above as...

$$I_5 = \int_s^t \frac{1}{\lambda}\left(\mathrm{Exp}\left\{ -\lambda\,(s+v) \right\} - \mathrm{Exp}\left\{ -\lambda\,(t+v) \right\} \right) \delta v = \frac{1}{\lambda}\int_s^t \mathrm{Exp}\left\{ -\lambda\,(s+v) \right\} \delta v - \frac{1}{\lambda}\int_s^t \mathrm{Exp}\left\{ -\lambda\,(t+v) \right\} \delta v \tag{32}$$

The solution to the left side of Equation (32) above is...

$$\int_s^t \mathrm{Exp}\left\{ -\lambda\,(s+v) \right\} \delta v = -\frac{1}{\lambda}\,\mathrm{Exp}\left\{ -\lambda\,(s+v) \right\} \Bigg[_{v=s}^{v=t} = \frac{1}{\lambda}\left(\mathrm{Exp}\left\{ -2\,\lambda\, s \right\} - \mathrm{Exp}\left\{ -\lambda\,(s+t) \right\} \right) \tag{33}$$

The solution to the right side of Equation (32) above is...

$$\int_s^t \mathrm{Exp}\left\{ -\lambda\,(t+v) \right\} \delta v = -\frac{1}{\lambda}\,\mathrm{Exp}\left\{ -\lambda\,(t+v) \right\} \Bigg[_{v=s}^{v=t} = \frac{1}{\lambda}\left(\mathrm{Exp}\left\{ -\lambda\,(s+t) \right\} - \mathrm{Exp}\left\{ -2\,\lambda\, t \right\} \right) \tag{34}$$

Using Equations (33) and (34) above the solution to Equation (32) above is...

$$I_5 = \frac{1}{\lambda}\left(\mathrm{Exp}\left\{ -2\,\lambda\, s \right\} - \mathrm{Exp}\left\{ -\lambda\,(s+t) \right\} \right) - \frac{1}{\lambda}\left(\mathrm{Exp}\left\{ -\lambda\,(s+t) \right\} - \mathrm{Exp}\left\{ -2\,\lambda\, t \right\} \right) \tag{35}$$

F. We want to solve the following integral product from Equation (18) above...

$$I = \mathrm{Exp}\left\{ 2\,\lambda\, s \right\} I_5 \tag{36}$$

Using Equation (35) above we can rewrite Equation (36) above as...

$$\mathrm{Exp}\left\{ 2\,\lambda\, s \right\} I_5 = \mathrm{Exp}\left\{ 2\,\lambda\, s \right\} \left[\frac{1}{\lambda}\left(\mathrm{Exp}\left\{ -2\,\lambda\, s \right\} - \mathrm{Exp}\left\{ -\lambda\,(s+t) \right\} \right) - \frac{1}{\lambda}\left(\mathrm{Exp}\left\{ -\lambda\,(s+t) \right\} - \mathrm{Exp}\left\{ -2\,\lambda\, t \right\} \right) \right] \tag{37}$$

The solution to Equation (37) above is...

$$\mathrm{Exp}\left\{ 2\,\lambda\, s \right\} I_5 = \frac{1}{\lambda^2}\left(1 - 2\,\mathrm{Exp}\left\{ -\lambda\,(t-s) \right\} + \mathrm{Exp}\left\{ -2\,\lambda\,(t-s) \right\} \right) \tag{38}$$

References

[1] Gary Schurman, *The Vasicek Interest Rate Process - The Stochastic Short Rate,* February, 2013.

[2] Gary Schurman, *Double Integral of a Minimum Function,* December, 2021.